

AUTOMORPHISMS ACTING ON THE LEFT-ORDERINGS OF A BI-ORDERABLE GROUP

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ABSTRACT. We generalize a result of Koberda [9], by showing that the natural action of the automorphism group on the space of left-orderings is faithful for all nonabelian bi-orderable groups G , as well as for a certain class of left-orderable groups that includes the braid groups. As a corollary we show that the action of $\text{Aut}(G)$ on ∂G is faithful whenever G is bi-orderable and hyperbolic, following the approach of [9]. We also analyze the action of the commensurator of G on its space of virtual left-orderings.

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1. INTRODUCTION

Let G be a group. We call a strict total ordering $<$ of the elements of G a *left-ordering* if $g < h$ implies $fg < fh$ for all $f, g, h \in G$. If G admits a left-ordering $<$ that is also right-invariant, in the sense that $g < h$ implies $gf < hf$ for all $f, g, h \in G$, then $<$ is a *bi-ordering* of G .

Each of these concepts can equivalently be defined in terms of positive cones. That is, given a left-ordering $<$ of G , we can identify $<$ with its positive cone

$$P = \{g \in G \mid g > 1\}$$

which is a subset of G satisfying:

- (1) $P \cdot P \subset P$
- (2) $P \sqcup P^{-1} \sqcup \{1\} = G$.

Conversely, given a subset $P \subset G$ satisfying (1) and (2), it determines a positive cone according to the prescription $g < h$ if and only if $g^{-1}h \in P$ for all $g, h \in G$. Bi-orderings may be similarly defined in terms of positive cones, but the positive cone of any bi-ordering must also satisfy a third condition, namely $gPg^{-1} \subset P$ for all $g \in G$.

We write $\text{LO}(G)$ for the set of all positive cones $P \subset G$ satisfying (i) and (ii) above, and, thinking of it as a subset of 2^G (equipped with the product topology) we endow $\text{LO}(G)$ with the subspace topology. Thus the open sets of $\text{LO}(G)$ are finite intersections of sets of the form

$$U_g = \{P \in \text{LO}(G) \mid g \in P\} \text{ and } U_g^c = \{P \in \text{LO}(G) \mid g^{-1} \in P\}.$$

We call $\text{LO}(G)$ the space of left-orderings of the group G . We similarly can define the space of bi-orderings of G , $\text{BiO}(G)$, by taking all positive cones P that satisfy the additional third

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condition of $gPg^{-1} \subset P$ for all $g \in G$. Topologizing $\text{BiO}(G)$ in the same way, we evidently have $\text{BiO}(G) \subset \text{LO}(G)$. Endowed with these topologies, both $\text{LO}(G)$ and $\text{BiO}(G)$ are compact spaces.

There is an action of G on $\text{LO}(G)$ defined by $g(P) = gPg^{-1}$. More generally, there is an action of $\text{Aut}(G)$ on $\text{LO}(G)$ by observing that $\phi(P)$ is again a positive cone for all $P \in \text{LO}(G)$ and $\phi \in \text{Aut}(G)$. The action of $\text{Aut}(G)$ on $\text{LO}(G)$ is an action by homeomorphisms. Since the positive cones which are fixed under conjugation correspond to the bi-orderings of G , there is also an action of $\text{Out}(G)$ on $\text{BiO}(G)$.

With the topological structure and group actions as above, $\text{LO}(G)$ has found many applications within the study of orderable groups (for example, it was used to show that every left-orderable group has finitely many or uncountably many left-orderings [10], and was used to demonstrate a connection between orderability and amenability [12]), though applications beyond the realm of orderability are few. In recent work Koberda provided an example of such an application, by showing that whenever G is a residually torsion-free nilpotent hyperbolic group, the natural action of $\text{Aut}(G)$ on ∂G is faithful [9]. This application relies on the following theorem, which was also extended in [13] by replacing $\text{Aut}(G)$ with the commensurator of G :

Theorem 1.1. [9, Theorem 1.1] *If G is a finitely generated residually torsion-free nilpotent group, then the natural action of $\text{Aut}(G)$ on $\text{LO}(G)$ is faithful.*

In this paper, we characterize the action of $\text{Aut}(G)$ on $\text{LO}(G)$ when G is a bi-orderable group. Recall that finitely-generated residually torsion-free nilpotent groups are bi-orderable, though the converse is not true. For example, Thompson's group F is bi-orderable, but not residually nilpotent since $[F, F]$ is a simple group [6, Section 1.2.4].

Note that for some bi-orderable groups, like \mathbb{Q}^k for all $k > 0$, we should not expect the action of $\text{Aut}(G)$ on $\text{LO}(G)$ to be faithful. For if $G = \mathbb{Q}^k$ then multiplication by a positive rational p/q in each coordinate of \mathbb{Q}^k can easily be seen to preserve all orderings of \mathbb{Q}^k . However, it turns out that these automorphisms of abelian groups are the only nontrivial automorphisms of bi-orderable groups which act trivially on the space of left-orderings. For an abelian group G and a fixed $p/q \in \mathbb{Q}$, we denote by $\tau_{p/q} : G \rightarrow G$ the automorphism satisfying $\tau_{p/q}(g^q) = g^p$ for all $g \in G$, when it exists. We prove:

Theorem 1.2. *Let G be a bi-orderable group.*

- (i) *If G is nonabelian then $\text{Aut}(G)$ acts faithfully on $\text{LO}(G)$.*
- (ii) *If G is abelian then the kernel of the action of $\text{Aut}(G)$ on $\text{LO}(G)$ contains precisely the automorphisms $\tau_{p/q}$, if any such automorphisms exist.*

Note that part (ii) of Theorem 1.2 already appears as [13, Proposition 4.3(2)]. We are also able to analyze the behaviour of the action of $\text{Aut}(G)$ on $\text{LO}(G)$ with respect to certain kinds of extensions.

Theorem 1.3. *Suppose that G is left-orderable and that*

$$1 \rightarrow K \rightarrow G \rightarrow \mathbb{Z} \rightarrow 1$$

is a short exact sequence of groups. Suppose that $\text{Aut}(K)$ acts faithfully on $\text{LO}(K)$. If conjugation by the generator of \mathbb{Z} preserves a left-ordering of K , then $\text{Aut}(G)$ acts faithfully on $\text{LO}(G)$.

Since bi-orderability is not preserved under extensions (even under extensions such as those in the statement of the theorem above), this allows us to create non-bi-orderable groups G for which $\text{Aut}(G)$ acts faithfully on $\text{LO}(G)$. See also Proposition 3.1.

As a corollary of Theorem 1.2 we can extend Koberda's result concerning the action of $\text{Aut}(G)$ on ∂G to all bi-orderable hyperbolic groups.

Corollary 1.4. *If G is a bi-orderable hyperbolic group, then $\text{Aut}(G)$ acts faithfully on ∂G .*

The proof of Corollary 1.4 is a combination of Theorem 1.2 and Proposition 4.1.

The paper is organized as follows. In Section 2 we provide additional background on left-orderings and bi-orderings of groups, and prove Theorem 1.2. In Section 3 we prove Theorem 1.3 and also study the braid groups B_n . In Section 4 show that the action of $\text{Aut}(G)$ on ∂G is faithful when G is hyperbolic and bi-orderable, and describe the action of $\text{Comm}(G)$ on $\text{VLO}(G)$ for all bi-orderable groups.

2. AUTOMORPHISMS OF BI-ORDERABLE GROUPS ACTING ON THE SPACE OF ORDERINGS

By insisting that the group G be bi-orderable, we allow ourselves some flexibility in creating new left-orderings of G . The orderings that we will create arise from considering the action of G on itself by conjugation, which is an order-preserving action if G is bi-ordered (see Lemma 2.2). With this line of reasoning we will create sufficiently many left-orderings to show that whenever $\phi \in \text{Aut}(G)$ and $\phi(g) \neq g$ for some $g \in G$, then there exists $P \in \text{LO}(G)$ that contains g but not $\phi(g)$. It follows that the action of ϕ on $\text{LO}(G)$ is nontrivial, because the positive cone P satisfies $\phi(P) \neq P$.

Recall that a subset $S \subset G$ is called *isolated* if $g^k \in S$ for some $k \in \mathbb{Z}$ implies that $g \in S$. The *isolator* of a subgroup H of G is the set

$$I(H) = \{g \in G \mid \text{there exists } k \in \mathbb{Z} \text{ such that } g^k \in H\}.$$

In general, $I(H)$ is not a subgroup. However, when H is abelian and G is bi-orderable, then $I(H)$ is an abelian subgroup. Essential in proving this fact is the following property of bi-orderable groups: In a bi-orderable group, when g^k and h^ℓ commute for some $k, \ell \in \mathbb{Z}$, then so do g and h . This fact will also be used several times in the proofs of this section.

When H is a rank one abelian subgroup of G , so is $I(H)$. If g is a nonidentity element of a bi-orderable group G , then we will denote the isolator of the cyclic subgroup $\langle g \rangle$ by $I(g)$ for

short. Thus $I(g)$ is always a rank one abelian group. We record the following fact for future use:

Lemma 2.1. *Let G be a group. If g, h are distinct elements of G , then either $I(g) = I(h)$ or $I(g) \cap I(h) = \{1\}$.*

Proof. Suppose there exists $f \in I(h) \cap I(g)$ where $f \neq 1$. Since $f \in I(g)$, there exist $n, m \in \mathbb{Z}$ such that $f^n = g^m$. But now $g^n \in I(h)$ and since $I(h)$ is isolated, g is also in $I(h)$ and $I(h) = I(g)$. \square

Recall that a subset S in a left-ordered group G is called *convex* with respect to a given left-ordering $<$ if $g, h \in S$ and $g < f < h$ implies $f \in S$. Of particular importance is the case when a subgroup C of a left-ordered group G is convex, as the convex subgroups of a left-ordering determine its structure in a sense described below. The convex subgroups of a left-ordered group G are ordered by inclusion. A subgroup is *relatively convex* if there exists a left-ordering relative to which it is convex.

Given a subgroup C of a left-ordered group G , the natural quotient ordering of the left cosets G/C is well-defined if and only if C is convex, in this case the natural left-action of G/C preserves the quotient ordering. Therefore we can think of the ordering of G as lexicographic: it is constructed via inclusion of the left-ordered subgroup C and via pullback of the natural ordering on the cosets G/C .

Consequently, if C is a convex subgroup of a left-ordered group, then the left-ordering of G may be altered by replacing the left-ordering of C with any left-ordering that we please. It follows that relative convexity is transitive, in the sense that if K is relatively convex in H , and H is relatively convex in G , then K is relatively convex in G . This fact is needed in the proof of the following lemma.

Lemma 2.2. [2, Lemma 2.4] *Suppose that G is a bi-orderable group, and that $g \in G$ is not the identity. Then $I(g)$ is relatively convex.*

Proof. Let G_i , $i = 1, 2$ denote two copies of the group G , and equip each copy with a given bi-ordering $<$. Create a total ordering of $G_1 \cup G_2$ using $<$ to order each G_i , and declare the elements of G_1 smaller than those of G_2 .

Now consider the action of G on $G_1 \cup G_2$ defined by conjugation on the elements of G_1 , and by left-multiplication on the elements of G_2 . This defines an effective, order-preserving action of G on the totally ordered set $G_1 \cup G_2$. Fix a nonidentity element $g \in G_1$ and well-order $G_1 \cup G_2$ so that g is smallest. Then using the action of G on $G_1 \cup G_2$ one may create a left-ordering of G in the standard way, relative to which $\text{Stab}_G(g) = C_G(g)$ is convex. Here, $C_G(g)$ denotes the centralizer of g in G (See [2, Proposition 2.3] or [3, Example 1.11 and Problem 2.16] for details of this construction). Now as $C_G(g)$ is bi-orderable, the centre $Z(C_G(g))$ is relatively convex in $C_G(g)$ by [1, Theorem 2.4]. Moreover, $I(g) \subset Z(C_G(g))$ since every element of $I(g)$ has some power which lies in $\langle g \rangle$, and thus commutes with all elements of $C_G(g)$. Since $I(g)$

is an isolated subgroup and $Z(C_G(g))$ is abelian, $I(g)$ is relatively convex in $Z(C_G(g))$. Thus $I(g)$ is relatively convex in G . \square

Proposition 2.3. *Suppose that G is a bi-orderable group, and that $\phi \in \text{Aut}(G)$. If there exists $g \in G$ such that $\phi(I(g)) \neq I(g)$, or if there exists $g \in G$ such that $\phi(g)^n = g^{-m}$ for some $m, n > 0$, then the action of ϕ on $\text{LO}(G)$ is nontrivial.*

Proof. Suppose there exists $g \in G$ such that $\phi(g)^n = g^{-m}$ for some $m, n > 0$. Consider an arbitrary positive cone $P \in \text{LO}(G)$. We can assume $g \in P$, if not we replace P by P^{-1} . Then $g \in P$ and $\phi(g) \notin P$, so we have $P \neq \phi(P)$.

Now suppose there exists g such that $I(g) \neq \phi(I(g))$, and note that $\phi(I(g)) = I(\phi(g))$. By Lemma 2.2 $I(g)$ is convex in some left-ordering of G with positive cone $P \in \text{LO}(G)$. Applying ϕ , one checks that $\phi(I(g)) = I(\phi(g))$ is convex relative to the ordering of G determined by $\phi(P)$.

To show that $\phi(P) \neq P$, we need only show that $I(g)$ is not convex relative to the ordering of G determined by $\phi(P)$. If it were, we would have either $I(g) \subset I(\phi(g))$ or $I(\phi(g)) \subset I(g)$, since convex subgroups are ordered by inclusion. By Lemma 2.1, either inclusion forces $I(g) = I(\phi(g)) = \phi(I(g))$, a contradiction. \square

Therefore, by Proposition 2.3, when G is a bi-orderable group and $\phi \in \text{Aut}(G)$ we know that ϕ acts nontrivially on $\text{LO}(G)$ unless ϕ satisfies:

$$(*) \quad \forall g \in \text{domain}(\phi) \exists n, m > 0 \text{ such that } \phi(g)^n = g^m.$$

We therefore investigate the existence of such automorphisms of bi-orderable groups.

Our lemmas below are stated in a slightly more general setting than needed in this section, as we will also be using them in our investigation of the action of $\text{Comm}(G)$ on $\text{VLO}(G)$ in Section 4.

Recall that when G is abelian, we denote by $\tau_{p/q} : G \rightarrow G$ the automorphism satisfying $\tau_{p/q}(g^q) = g^p$ for all $g \in G$, when it exists. More generally, if H_1, H_2 are finite index abelian subgroups of a group G , we denote by $\tau_{p/q} : H_1 \rightarrow H_2$ the isomorphism satisfying $\tau_{p/q}(g^q) = g^p$ for all $g \in H_1$, when it exists.

Lemma 2.4. *Suppose G is a bi-orderable group with finite index torsion-free abelian subgroups H_1, H_2 , and $\phi : H_1 \rightarrow H_2$ is an isomorphism satisfying $(*)$. Then there exist $p, q > 0$ such that $\phi(g)^q = g^p$ for all $g \in H_1$, so that $\phi = \tau_{p/q}$.*

Proof. This lemma is essentially Case 2 of the proof of [13, Proposition 4.3]. Here is an alternative proof. Assume $\phi : H_1 \rightarrow H_2$ satisfies $(*)$ and that H_1 is torsion free abelian. Let $g, h \in H_1$ and suppose $\phi(g)^m = g^n$ and $\phi(h)^\ell = h^k$ for some $k, \ell, m, n > 0$. By uniqueness of roots, we may assume that $\gcd(m, n) = \gcd(k, \ell) = 1$, we wish to show that $m = \ell$ and $n = k$. If $I(g) = I(h)$ then the result follows by applying ϕ to a common power of g and h which lies in

H_1 , such a common power exists since $|G : H_1|$ is finite. So suppose $I(g) \neq I(h)$, and therefore $I(g) \cap I(h) = \{1\}$ by Lemma 2.1.

Considering $g^m h^\ell$, we see that $\phi(g^m h^\ell) = g^n h^k \in I(g^m h^\ell)$, so there exist relatively prime $s, t > 0$ such that $(g^m h^\ell)^s = (g^n h^k)^t$. Since H_1 is abelian $g^{ms-nt} = h^{tk-sl}$, and since both are in $I(g) \cap I(h)$, both are equal to 1. Since $\gcd(m, n) = \gcd(s, t) = 1$, from $ms - nt = 0$ we find $m = t$ and $s = n$. Similarly from $tk - sl$ we find $t = \ell$ and $k = s$, so we are done. \square

Lemma 2.5. *Suppose G is a bi-orderable group with finite index subgroups H_1, H_2 , that $\phi : H_1 \rightarrow H_2$ is an isomorphism satisfying (*), and that ϕ is not the identity. Then for every $g \in H_1$ there exist $p, q > 0$ such that $\phi(g)^q = g^p$ where $p \neq q$.*

Proof. Since ϕ is not the identity there exists $g \in H_1$ with $\phi(g) \neq g$, say $\phi(g)^s = g^t$ with $s \neq t$ (necessarily $s \neq t$ since G is bi-orderable). Now let $h \in G$ be given. By (*) there exists $n, m > 0$ such that $\phi(h)^n = h^m$. If $n = m$ then $\phi(h) = h$ since G is bi-orderable. But $\phi(g^s h) = g^t h$, so $g^t h \in I(g^s h)$. But then $(g^t h)(h^{-1} g^{-s}) = g^{t-s} \in I(g^s h)$. Therefore $g \in I(g^s h)$, and so $h \in I(g^s h)$, and $I(g) = I(h)$. Now since $I(g)$ is abelian we may apply Lemma 2.4 to the restriction isomorphism $\phi|_{I(g)} : I(g) \rightarrow I(g)$ arising from ϕ . We conclude that $n = s$ and $m = t$, contradicting the fact that $n \neq m$. Thus $n \neq m$. \square

Note that we can improve the conclusion of the previous lemma, by using uniqueness of roots in a bi-orderable group to show that p, q exist with $\gcd(p, q) = 1$. However this is not needed for our purposes.

Lemma 2.6. *Suppose G is a bi-orderable group with finite index subgroups H_1, H_2 and that $\phi : H_1 \rightarrow H_2$ is an isomorphism satisfying (*). Let $g, h \in H_1$ be given and suppose that $\phi(g)^m = g^n$ and $\phi(h)^\ell = h^k$. Then*

$$g^{n-m} h g^{m-n} \in I(h) \text{ and } h^{k-\ell} g h^{\ell-k} \in I(g).$$

Proof. By symmetry, it suffices to show only $g^{n-m} h g^{m-n} \in I(h)$. First, notice that $\phi(f) \in I(f)$ for all $f \in H_1$ by (*). Therefore $\phi(g^m h^\ell g^{-m}) = g^n h^k g^{-n} \in I(g^m h g^{-m})$, and since $I(g^m h g^{-m})$ is isolated we conclude $g^n h g^{-n} \in I(g^m h g^{-m})$. Next, notice that if $x \in I(h)$ then $g^i x g^{-i} \in I(g^i h g^{-i})$ for all $i \in \mathbb{Z}$, and thus $g^{n-m} h g^{m-n} \in I(h)$. \square

Lemma 2.7. *Suppose G is a bi-orderable group with finite index subgroups H_1, H_2 , that $\phi : H_1 \rightarrow H_2$ is an isomorphism satisfying (*), and that ϕ is not the identity. Then H_1 is abelian.*

Proof. Let $g, h \in H_1$ be given. If $I(g) = I(h)$ then g and h commute. Thus we assume $I(g) \neq I(h)$. By Lemma 2.5 there exist $m, n > 0$ and $k, \ell > 0$ with $m \neq n$ and $k \neq \ell$ such that $\phi(g)^m = g^n$ and $\phi(h)^\ell = h^k$. Consider $h^{k-\ell} g^{n-m} h^{\ell-k} g^{m-n}$. On one hand, we have $h^{k-\ell} g^{n-m} h^{\ell-k} g^{m-n} = (h^{k-\ell} g^{n-m} h^{\ell-k}) \cdot g^{m-n} \in I(g)$, since it is a product of elements of $I(g)$ (here we use Lemma 2.6). On the other hand, $h^{k-\ell} \cdot (g^{n-m} h^{\ell-k} g^{m-n}) \in I(h)$ by similar reasoning. By Lemma 2.1 $I(g) \cap I(h) = \{1\}$ and so $h^{k-\ell} g^{n-m} h^{\ell-k} g^{m-n} = 1$. But this means the nontrivial powers $h^{k-\ell}$ and g^{n-m} commute, so h and g commute since G is bi-orderable. Thus H_1 is abelian. \square

Proof of Theorem 1.2. Let G be a bi-orderable group and let $\phi \in \text{Aut}(G)$ be nontrivial. If G is nonabelian, then by Lemma 2.7 ϕ cannot satisfy (*). By Proposition 2.3 ϕ acts nontrivially on $\text{LO}(G)$, so the action of $\text{Aut}(G)$ on $\text{LO}(G)$ is faithful.

If G is abelian, and if ϕ does not satisfy (*), then Proposition 2.3 tells us that ϕ acts nontrivially on $\text{LO}(G)$. If ϕ does satisfy (*), then Lemma 2.4 tells us that $\phi = \tau_{p/q}$ for some $p/q \in \mathbb{Q}$. It is easy to see that in this case, ϕ acts trivially on $\text{LO}(G)$. Thus the kernel of the action of $\text{Aut}(G)$ on $\text{LO}(G)$ consists exactly of the automorphisms $\tau_{p/q}$. \square

3. NON-BI-ORDERABLE GROUPS

For certain classes of left-orderable groups, it is sometimes sufficient to examine the action of $\text{Aut}(G)$ on a small subset of $\text{LO}(G)$ (perhaps even a finite subset) in order to determine that the action is faithful.

Recall that a left-ordering of G is *discrete* if there is a smallest positive element. If $\phi : G \rightarrow G$ is an automorphism, and if P is the positive cone of a discrete left-ordering with smallest positive element $g \in G$, then $\phi(P)$ is the positive cone of a discrete left-ordering whose smallest positive element is $\phi(g)$. Thus if $g \neq \phi(g)$, then $P \neq \phi(P)$. We apply this idea in the following proposition.

Proposition 3.1. *Suppose that G is a left-orderable group with generators $\{g_i\}_{i \in I}$, and that for each $i \in I$ there exists $P_i \in \text{LO}(G)$ which is the positive cone of a discrete left-ordering with g_i as smallest positive element. Then $\text{Aut}(G)$ acts faithfully on $\text{LO}(G)$.*

Proof. If $\phi : G \rightarrow G$ is a nontrivial automorphism, then there exists a generator g_i such that $\phi(g_i) \neq g_i$. But then $\phi(P_i) \neq P_i$, so that ϕ acts nontrivially on $\text{LO}(G)$. \square

Example 3.2. Recall the Artin presentation of braid group B_n is given by

$$B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \mid \begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| > 1 \\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \text{ if } |i - j| = 1 \end{array} \right\rangle.$$

By Dehornoy, the braid groups B_n are left orderable for all n , as is the braid group B_∞ [4]. The *Dehornoy ordering* of B_n is a left-ordering that is defined in terms of representative words of braids as follows: A word w in the generators $\sigma_1, \dots, \sigma_{n-1}$ is called *i*-positive (respectively *i*-negative) if w contains at least one occurrence σ_i , no occurrence of $\sigma_1, \dots, \sigma_{i-1}$, and every occurrence of σ_i has positive (respectively negative) exponent. A braid $\beta \in B_n$ is called *i*-positive (respectively *i*-negative) if it admits a representative word w in the generators $\sigma_1, \dots, \sigma_{n-1}$ that is *i*-positive (respectively *i*-negative). The Dehornoy ordering of the braid group B_n is the ordering whose positive cone P_D is the set of all braids $\beta \in B_n$ that are *i*-positive for some i . Using $sh^{n-j} : B_j \rightarrow B_n$ to denote the shift homomorphism sending σ_i to σ_{i+j} , the convex subgroups of B_n are $sh^{n-j}(B_j) = \langle \sigma_{n-j+1}, \dots, \sigma_{n-1} \rangle \subset B_n$ [5], in particular the Dehornoy ordering is discrete with smallest positive element σ_{n-1} .

We can also define a related left-ordering as follows: a word w in generators $\sigma_1, \dots, \sigma_{n-1}$ is called *i*-reverse positive, if it has no occurrence of $\sigma_{i+1}, \dots, \sigma_{n-1}$, and every occurrence of σ_i has

positive exponent. Now similar to Dehornoy ordering, define an ordering $<'_D$ on B_n , whose positive cone P'_D consists of all braids $\beta \in B_n$ that are i -reverse positive for some i .

It is a straightforward check that $<'_D$ is also a discrete ordering of B_n , with σ_1 as its least positive element. Moreover, the convex subgroups of B_n with respect to $<'_D$ are exactly the subgroups $B_j = \langle \sigma_1, \dots, \sigma_{j-1} \rangle \subset B_n$ for $1 \leq j \leq n$.

Now given any i where $1 \leq i \leq n-1$, we can construct a left ordering $<_i$ on B_n with σ_i as its least positive element. First, we left-order B_n with $<'_D$. Since B_{i+1} is convex with respect to P'_D , we can replace the left ordering $<'_D$ on B_{i+1} with the left ordering of $<_D$. Denote the resulting ordering of B_n by $<_i$. By construction, $<_i$ is a discrete ordering with σ_i as its least positive element. Based on this construction and Proposition 3.1, $\text{Aut}(B_n)$ acts faithfully on $\text{LO}(B_n)$.

This same construction can also be used to produce a left-ordering of B_∞ with σ_i as smallest positive element for all $i \geq 1$. Thus $\text{Aut}(B_\infty)$ acts faithfully on $\text{LO}(B_\infty)$ as well. \square

If K and H are bi-orderable groups and

$$1 \rightarrow K \rightarrow G \rightarrow H \rightarrow 1$$

is a short exact sequence, then G can be lexicographically bi-ordered if and only if there exists a bi-ordering of K whose positive cone is invariant under the conjugation action of H . By relaxing this condition, we are able to create groups which are *not* bi-orderable, but for which the automorphism group acts faithfully on the space of left-orderings.

Theorem 3.3. *Suppose that G is left-orderable and that*

$$1 \rightarrow K \rightarrow G \rightarrow \mathbb{Z} \rightarrow 1$$

is a short exact sequence of groups. Suppose that $\text{Aut}(K)$ acts faithfully on $\text{LO}(K)$. If conjugation by the generator of \mathbb{Z} preserves a left-ordering of K , then $\text{Aut}(G)$ acts faithfully on $\text{LO}(G)$.

Proof. Suppose that $\phi : G \rightarrow G$ is a nontrivial automorphism. If $\phi(K) \neq K$, choose $g \in K$ with $\phi(g) \notin K$. Then by choosing signs appropriately, we may use the given short exact sequence to construct a positive cone $P \subset G$ for which $g \in P$ while $\phi(g) \notin P$. Thus $\phi(P) \neq P$.

On the other hand, suppose that $\phi(K) = K$. If there exists $k \in K$ for which $\phi(k) \neq k$, then we know there is a positive cone $P_K \in \text{LO}(K)$ for which $\phi(P_K) \neq P_K$ since $\text{Aut}(K)$ acts faithfully on $\text{LO}(K)$. Using the given short exact sequence we may extend P_K to a positive cone $P \subset G$ satisfying $\phi(P) \neq P$.

Last, suppose that $\phi(k) = k$ for all $k \in K$, and choose $t \in G$ which maps to the generator of \mathbb{Z} . Equip K with a positive cone P_K that is preserved by conjugation by t , and proceed as in [16, Lemma 3.4]. Note that every $g \in G$ can be written uniquely as kt^n for some $n \in \mathbb{Z}$ and $k \in K$, and since ϕ is nontrivial and satisfies $\phi(k) = k$ for all $k \in K$ it follows that $\phi(t) \neq t$. Construct a positive cone $P \subset G$ as follows: an element kt^n is in P if $k \in P_K$ or $k = 1$ and $n > 0$. Then

P clearly satisfies $P \cup P^{-1} = G \setminus \{1\}$ and $P \cap P^{-1} = \emptyset$. Moreover if kt^n and $k't^m$ are both in P , then so is $kt^n k't^m = k(t^n k' t^{-n}) t^{m+n}$ since conjugation by t preserves P_K . One can easily verify that the subgroup $\langle t \rangle$ is convex relative to the ordering of G determined by P , so that P determines a discrete ordering of G with t as smallest positive element. The positive cone $\phi(P)$ will determine a left-ordering of G with $\phi(t)$ as smallest positive element. As $\phi(t) \neq t$, we conclude that $\phi(P) \neq P$. \square

If K is a bi-orderable group, automorphisms $\phi : K \rightarrow K$ which preserve a left-ordering of K but not a bi-ordering are likely quite common. However, there is little in the literature dealing with automorphism-invariant left-orderings, as the focus has primarily been on automorphism-invariant bi-orderings [14, 15, 11].

Here is one example of how an automorphism-invariant left-ordering (which is not a bi-ordering) may arise, which we use to illustrate an application of Theorem 3.3.

Example 3.4. Set $K = \mathbb{Q}^2 \rtimes \mathbb{Z}$ where the conjugation action of \mathbb{Z} on \mathbb{Q}^2 is by the matrix $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$. Then K is bi-orderable, since the action of A preserves the bi-ordering of \mathbb{Z}^2 defined by $(a, b) > (0, 0)$ if and only if $(a, b) \cdot (\sqrt{2}, 1) = b + \sqrt{2}a > 0$. In fact, since the eigenvectors of A are $\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$ with positive and negative eigenvalues respectively, the ordering described above (and its opposite) are the only orderings of \mathbb{Q}^2 preserved by A . Thus K is bi-orderable and nonabelian, so by Theorem 1.2 $\text{Aut}(K)$ acts faithfully on $\text{LO}(K)$.

Now we define $G = K \rtimes \mathbb{Z}$ where the action of the generator of \mathbb{Z} on an element of K is $((a, b), c) \mapsto (-A(a, b)^T, c)$. The action of $-A$ on the subgroup $\mathbb{Q}^2 \subset K$, having the same eigenvectors as A but with eigenvalues of opposite sign, preserves only the ordering defined by $(a, b) > (0, 0)$ if and only if $(a, b) \cdot (-\sqrt{2}, 1) = b - \sqrt{2}a > 0$, and its opposite. Using this ordering on \mathbb{Q}^2 , and lexicographically ordering K using the short exact sequence $1 \rightarrow \mathbb{Q}^2 \rightarrow K \rightarrow \mathbb{Z} \rightarrow 1$, we arrive at a left-ordering of K preserved by the action of the generator of \mathbb{Z} .

We conclude $\text{Aut}(G)$ will act faithfully on $\text{LO}(G)$ by Theorem 3.3.

Note that G is left-orderable by a straightforward short exact sequence argument, but is not bi-orderable since the actions of A and $-A$ on $\mathbb{Q}^2 \subset K$ do not preserve a common ordering, so Theorem 1.2 does not apply. Proposition 3.1 also cannot apply to G since any generator of $\mathbb{Q}^2 \subset G$ cannot be the smallest positive element of a left-ordering of G . \square

Despite these extensions and examples, one cannot hope to replace “bi-orderable” in Theorem 1.2 with either the weaker condition of local indicability or the condition that G admit an ordering that is recurrent for every cyclic subgroup (See [12] for more information on recurrent orderings). Koberda points out that for the Klein bottle group, $K = \langle x, y \mid xyx^{-1} = y^{-1} \rangle$, the action of $\text{Aut}(K)$ on $\text{LO}(K)$ is not faithful. Yet K is both locally indicable and admits recurrent orderings, as it only has four left-orderings.

4. APPLICATIONS AND GENERALIZATIONS

The action of $\text{Aut}(G)$ on $\text{LO}(G)$ is connected to the action of $\text{Aut}(G)$ on ∂G by the following theorem. Though not stated in full generality in [9], the proof below appears there as part of the proof of [9, Theorem 1.2]. As it is relatively short, we repeat it here for the reader's convenience. For background and further information on hyperbolic groups, see [9, 7, 8].

Proposition 4.1. *If G is a left-orderable hyperbolic group and $\text{Aut}(G)$ acts faithfully on $\text{LO}(G)$, then it acts faithfully on ∂G .*

Proof. Recall that for each element $g \in G$, there are two distinct points in the boundary ∂G defined by $x_g = \lim_{n \rightarrow \infty} g^n$ and $y_g = \lim_{n \rightarrow \infty} g^{-n}$. Moreover, given $g, h \in G$ if $\langle g, h \rangle$ is not a virtually cyclic group, then g and h determine distinct points on the boundary.

Choose a nontrivial automorphism $\phi \in \text{Aut}(G)$, $g \in G$ and $P \in \text{LO}(G)$ such that $g \in P$ and $\phi(g) \notin P$ (and thus $\phi(P) \neq P$). Since we cannot have $g^k = \phi(g)^\ell$ for some $k, \ell > 0$, there are two cases. Recall we defined $I(g)$ in Section 2 to be the isolator of the cyclic subgroup $\langle g \rangle$.

Case 1. $\phi(g) \in I(g)$ and there exists $k, \ell > 0$ such that $g^{-k} = \phi(g)^\ell$. In this case, observe that $x_g = \lim_{n \rightarrow \infty} (g^\ell)^n$, so that $\phi(x_g) = \lim_{n \rightarrow \infty} \phi(g^\ell)^n = \lim_{n \rightarrow \infty} (g^{-k})^n = y_g$, so that ϕ acts nontrivially on ∂G .

Case 2. $\phi(g) \notin I(g)$. Then $\phi(g)$ and g do not generate a virtually cyclic subgroup, so x_g and $\phi(x_g) = x_{\phi(g)}$ are distinct. Thus ϕ acts nontrivially on ∂G . \square

Consequently, by applying Theorem 1.2, we arrive at Corollary 1.4. If G is hyperbolic and satisfies the hypotheses of Theorem 3.3 or Proposition 3.1, then $\text{Aut}(G)$ acts faithfully on ∂G , too. However it seems difficult to construct a hyperbolic group G satisfying the hypotheses of either result.

There are also two natural generalizations one may consider, both developed by Witte Morris in [13]. First, one may replace the automorphism group with the commensurator group $\text{Comm}(G)$ of G . Recall that a *commensuration* of a group G is an isomorphism $\phi : H_1 \rightarrow H_2$ of finite index subgroups $H_i \subset G$. Two commensurations $\phi : H_1 \rightarrow H_2$ and $\phi' : H'_1 \rightarrow H'_2$ are equivalent if there exists a finite index subgroup $H \subset H_1 \cap H'_1$ such that $\phi|_H = \phi'|_H$. The set of equivalence classes of commensurations forms the *commensurator group* $\text{Comm}(G)$ of G .

Witte Morris points out that for torsion free locally nilpotent groups, $\text{Comm}(G)$ acts naturally on $\text{LO}(G)$. This follows from an application of Koberda's theorem (Theorem 1.1), and the fact that for every subgroup H of a torsion-free locally nilpotent group G , the restriction map $r : \text{LO}(G) \rightarrow \text{LO}(H)$ is surjective. When G is a bi-orderable group, the restriction $r : \text{LO}(G) \rightarrow \text{LO}(H)$ is not a surjective map in general, so this generalization is not possible in our setting.

However, using the restriction map $r : \text{LO}(G) \rightarrow \text{LO}(H)$ for each finite index subgroup $H \subset G$, one can define the space of *virtual* left-orderings of G as the limit

$$\text{VLO}(G) = \varinjlim \text{LO}(H),$$

where the limit is over all finite-index subgroups H of G [13]. When $P \in \text{LO}(H)$ and H is a finite index subgroup of G , we will denote the corresponding element of $\text{VLO}(G)$ by $[P]$. Then $\text{Comm}(G)$ naturally acts on $\text{VLO}(G)$: for each commensuration $\phi : H_1 \rightarrow H_2$ and each positive cone $P \in \text{LO}(H)$, set $\phi([P]) = [\phi(P \cap H_1)]$. It is straightforward to check that this definition respects the necessary equivalence relations.

Lemma 4.2. *Let G be a left-orderable group and $\phi : H_1 \rightarrow H_2$ a commensuration of G where H_1 is abelian. If $\phi = \tau_{p/q}$ for some $p/q \in \mathbb{Q}$ then the element of $\text{Comm}(G)$ represented by ϕ acts trivially on $\text{VLO}(G)$.*

Proof. Suppose that H is a finite index subgroup and $P \subset H$ is the positive cone of a left-ordering. Consider $P \cap H_1$ and $\phi(P \cap H_1)$. The first is the positive cone of a left-ordering of $H \cap H_1$, the second is the positive cone of a left-ordering of $H \cap H_2$. Using the fact that ϕ satisfies (*), one can show that these orderings agree on the finite index subgroup $H \cap H_1 \cap H_2$ so that $[P] = [\phi(P \cap H_1)]$, and thus ϕ acts trivially on $\text{VLO}(G)$. \square

Theorem 4.3. *Let G be a bi-orderable group.*

- (i) *If G is not virtually abelian then $\text{Comm}(G)$ acts faithfully on $\text{VLO}(G)$.*
- (ii) *If G is virtually abelian then the kernel of the action of $\text{Comm}(G)$ on $\text{VLO}(G)$ contains precisely the elements represented by commensurations $\tau_{p/q} : H_1 \rightarrow H_2$, if any such commensurations exist.*

Proof. First suppose that G is not virtually abelian, and let $\phi : H_1 \rightarrow H_2$ be a nontrivial commensuration of G . By Lemma 2.7, ϕ cannot satisfy (*) since H_1 is not abelian. Thus there exists $g \in H_1$ such that $\phi(g) \neq g$ and either $\phi(g)^n = g^{-m}$ for some $m, n > 0$ or $I(g) \neq I(\phi(g))$. In either case we can construct a left-ordering of G with positive cone P satisfying $g \in P$ and $\phi(g) \notin P$ using arguments identical to those in the proof of Lemma 2.3. Then $[P] \neq [\phi(P \cap H_1)]$, so (the class of) $\phi : H_1 \rightarrow H_2$ acts nontrivially on $\text{VLO}(G)$.

On the other hand, suppose G is virtually abelian, and let $\phi : H_1 \rightarrow H_2$ be a nontrivial commensuration of G . If ϕ does not satisfy (*), then an argument identical to the previous paragraph shows that the class of ϕ acts nontrivially on $\text{VLO}(G)$. On the other hand, if ϕ does satisfy (*), then $\phi = \tau_{p/q}$ for some $p/q \in \mathbb{Q}$ by Lemma 2.4. In this case, $\phi = \tau_{p/q}$ acts trivially on $\text{VLO}(G)$ by Lemma 4.2. \square

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